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ON THE LOCALIZATION PROPERTIES OF MULTIPLICATIVE HYPERELASTO-PLASTIC CONTINUA WITH STRONG DISCONTINUITIES

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Abstract—The objective of this work is to examine the large strain localization properties of *hyperelasto-plastic* materials which are based on the *multiplicative* decomposition of the deformation gradient. Thereby, the case of *strong discontinuities* is investigated. To this end, first an explicit expression for the spatial tangent operator is given, taking into account *anisotropic* as well as *nonassociated* material behaviour. Then the structure of a *regularized discontinuous* velocity gradient is elaborated and discussed in detail. Based on these two results, the localization condition is derived with special emphasis on the loading conditions inside and outside an anticipated localization band. Thereby, the intriguingly simple structure of the tangent operator, which resembles the structure of the geometrically linear theory, is extensively exploited. This similarity carries over to the general representation for the critical hardening modulus which is exemplified for *isotropic* materials. As a result, *analytical* solutions are available under the assumption of small elastic strains, which is justified for metals. Finally, examples are given for the special case of the *associated* von Mises flow rule. To this end, the critical localization direction and the critical hardening modulus are investigated with respect to the amount of *finite elastic* strain within different modes of homogeneous *elasto-plastic* deformations. © 1997, Elsevier Science Ltd. All rights reserved.

NOTATION

Material tensors defined in $\mathscr{B}_0$:

$\mathbf{C} = \mathbf{F}^t \cdot \mathbf{F}$	right Cauchy–Green deformation
$\mathbf{L} = \mathbf{F}^{-1} \cdot \mathbf{l} \cdot \mathbf{F} = \mathbf{F}^{-1} \cdot \dot{\mathbf{F}}$	Lagrange velocity gradient
$\mathbf{N}$	material normal to surface $\mathrm{d}A$ at $\mathbf{X}$
$\mathbf{\Sigma}_2 = \mathbf{\Sigma}_1 \cdot \mathbf{F}^{-t} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-t}$	2. Piola–Kirchhoff stress
$\mathbf{T} = \mathbf{F}^t \cdot \mathbf{\Sigma}_1^t = \mathbf{F}^t \cdot \boldsymbol{\tau}^t \cdot \mathbf{F}^{-t}$	Mandel stress

Spatial tensors defined on $\mathscr{B}$:

$\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^t$	left Cauchy–Green deformation
$\mathbf{l} = \mathbf{F} \cdot \mathbf{L} \cdot \mathbf{F}^{-1} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}$	Euler velocity gradient
$\mathbf{n}$	spatial normal to surface $\mathrm{d}a$ at $\mathbf{x}$
$\mathbf{t} = J^{-1} \boldsymbol{\tau}^t \cdot \mathbf{n}$	true traction
$\mathbf{t}_1 = \mathbf{\Sigma}_1^t \cdot \mathbf{N}$	nominal traction
$\boldsymbol{\tau} = \mathbf{F} \cdot \mathbf{\Sigma}_1 = \mathbf{F} \cdot \mathbf{\Sigma}_2 \cdot \mathbf{F}^t$	Kirchhoff stress

Two point tensors defined on $\mathscr{B}_0$ and $\mathscr{B}$:

$\mathbf{F}$	deformation gradient with $\det \mathbf{F} = J$
$\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 \cdot \mathbf{F}^t = \mathbf{F}^{-1} \cdot \boldsymbol{\tau}$	1. Piola–Kirchhoff or nominal stress

Objects with a superposed bar ($\bar{\bullet}$) are 'material' tensors defined on the intermediate configuration $\bar{\mathscr{B}}$. Please observe that we will denote by $\bar{\mathbf{N}}$ the normal to yield condition $\partial_{\mathbf{T}}\Phi$ on $\bar{\mathscr{B}}$ and not a normal to a surface.

Our notation employs the following dyadic products of second order tensors $\mathbf{a}$ and $\mathbf{b}$ when expressed in index notation referred to a Cartesian co-ordinate system

$$[\mathbf{a} \otimes \mathbf{b}]_{ijkl} = [\mathbf{a}]_{ij}[\mathbf{b}]_{kl} \quad \text{with} \quad [\mathbf{a} \otimes \mathbf{b}] : \mathbf{c} = \mathbf{a}[\mathbf{c} : \mathbf{b}] \quad \text{and} \quad \mathbf{c} : [\mathbf{a} \otimes \mathbf{b}] = [\mathbf{a} : \mathbf{c}]\mathbf{b}$$

$$[\mathbf{a} \overline{\otimes} \mathbf{b}]_{ijkl} = [\mathbf{a}]_{ik}[\mathbf{b}]_{jl} \quad \text{with} \quad [\mathbf{a} \overline{\otimes} \mathbf{b}] : \mathbf{c} = \mathbf{a} \cdot \mathbf{c} \cdot \mathbf{b}^t \quad \text{and} \quad \mathbf{c} : [\mathbf{a} \overline{\otimes} \mathbf{b}] = \mathbf{a}^t \cdot \mathbf{c} \cdot \mathbf{b}$$

$$[\mathbf{a} \underline{\otimes} \mathbf{b}]_{ijkl} = [\mathbf{a}]_{il}[\mathbf{b}]_{jk} \quad \text{with} \quad [\mathbf{a} \underline{\otimes} \mathbf{b}] : \mathbf{c} = \mathbf{a} \cdot \mathbf{c}^t \cdot \mathbf{b}^t \quad \text{and} \quad \mathbf{c} : [\mathbf{a} \underline{\otimes} \mathbf{b}] = \mathbf{b}^t \cdot \mathbf{c}^t \cdot \mathbf{a}.$$

1. INTRODUCTION

A failure phenomenon, which is frequently observed in laboratory experiments as well as in nature, is the *localization* of inelastic deformations within narrow bands. To investigate such localization, which forms a precursor to fracture, is a challenging problem from the viewpoints of constitutive modelling as well as of the choice of computational strategy.

The *classical* approach is to consider a band that is trapped between two material surfaces, across which the material or the spatial velocity gradient is discontinuous (weak discontinuity), Rice (1976), Thomas (1961), Rudnicki and Rice (1975). Analogous arguments in the context of planar acceleration waves in solids are found in the early work of Hadamard (1903) and in the contributions by Hill (1962) and Mandel (1962, 1966). Moreover, the traditional analysis relies on the assumption of a linear comparison solid in the sense of Hill (1958). Discontinuous bifurcation is then reflected by a singularity of the spatial localization tensor. As an example, we refer to the small strain analysis of an orthotropic yield condition by Steinmann *et al.* (1994). Within the context of quasi-static boundary value problems, the first occurrence of the singularity of the localization tensor is referred to as the *loss of ellipticity*, which is synonymous with the appearance of real characteristics associated with governing equations of the hyperbolic type. Likewise, within the context of dynamic initial boundary value problems, the singularity of the localization tensor is connected with the *loss of hyperbolicity*.

The *alternative*, and kinematically more general, approach pursued in this paper is to assume that the localization zone is the result of a *regularized* discontinuity in the nonlinear deformation map, or displacement field, itself (strong discontinuity). Moreover, since no linear comparison solid is assumed in this approach, different loading and unloading scenarios are analysed with respect to localization. This results in a much broader interpretation of the spectral properties of the localization tensor. So far in the literature, the conditions for localization within this framework have been established merely for the small strain approximation, e.g., Ottosen and Runesson (1991), and localization capturing FE-strategies have been proposed by Larsson *et al.* (1993), Simo *et al.* (1993) and Larsson and Runesson (1996), Larsson *et al.* (1996). It is emphasized that, as long as the necessary conditions for the *onset* of localization are concerned, the difference between the two approaches is quite subtle in practice, whereas the impact on the computational algorithm may be significant. Therefore, the results of this work are useful for future numerical developments.

For the constitutive framework, we elaborate in this paper on the concept of a *regularized* displacement discontinuity within the context of multiplicative *hyperelasto-plastic* continua at large strains. Although the concept of a multiplicative decomposition of the deformation gradient goes back to early works by Kröner (1960), Lee (1969), Teodosiu and Sidoroff (1976) and Mandel (1972) it is not until the contributions by, e.g., Simo and Ortiz (1985), Simo (1988), that it has won widespread acceptance over the more traditional additive type *hypoelasto-plastic* models. The formulation and numerical treatment of multiplicative *hyperelasto-plasticity* has progressed extensively in the recent works of Moran *et al.* (1990), Cuitiño and Ortiz (1992), Simo (1992), Simo and Miehe (1992), Miehe and Stein (1992), Miehe (1994). One significant advantage is that the formulation is based on the sound thermodynamic principle of *nonnegative* energy dissipation. Micromechanically, the multiplicative decomposition is strongly motivated by the kinematics of single crystals where the deformation is decomposed into dislocation movement along fixed slip systems followed by an elastic distorsion of the crystal lattice. Nevertheless, the multiplicative decomposition is presently widely accepted as a general concept for large strain *hyperelasto-plasticity* of a broad class of materials.

The paper is organised as follows: first, a summary is given of the major steps towards the establishment of the so-called first Euler *hyperelasto-plastic* tangent operator, which is used in the *spatial setting* for the constitutive relations. In passing, we note some new observations concerning the structure of this tensor. Next, we introduce the concept of a *regularized* strong discontinuity across a material surface attached to the body, i.e., convecting with the deformation. Then we discuss the general localization condition in terms of traction continuity, which is followed by an investigation of how *elastic unloading* vs *plastic loading* will affect the localization assessment. For the sake of demonstration, we establish explicit equations for the example of compressible Neo–Hooke hyperelasticity coupled to arbitrary isotropic yield conditions, from which the critical hardening modulus and the corresponding critical localization directions can be computed. In particular, we show the remarkable result in the case of *small elastic* strains, which may be superimposed on *finite plastic* deformations, that the analytical results obtained by Ottosen and Runesson (1991) are directly applicable. Finally, the analysis of typical large strain examples of uniaxial tension, pure and simple shear illustrate the theory.

2. BASIC STRESS RATES AND TANGENT OPERATORS

To set the stage for the subsequent developments, we briefly review some essential relations of *geometrically nonlinear* continuum descriptions. As usual, the nonlinear deformation map $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}) : \mathscr{B}_0 \to \mathbb{R}^3$ maps particle positions $\mathbf{X}$ in the reference configuration $\mathscr{B}_0$ to their actual position $\mathbf{x}$ in the deformed configuration $\mathscr{B}$. Then $\mathbf{F} = \nabla_X \boldsymbol{\varphi}$ with $J = \det \mathbf{F} > 0$, denotes the deformation gradient defining a linear map of elements in the tangent space $T\mathscr{B}_0$ into elements of the tangent space $T\mathscr{B}$. The pertinent strain and stress measures, that will be used subsequently, are summarized in the notations for brevity and clarity. Here, we will mainly discuss some useful stress rates and the corresponding tangent stiffness operators. Associated with the transformations between the reference and the spatial configurations $\mathscr{B}_0$ and $\mathscr{B}$, i.e., from the *pull-back* operations applied to the Kirchhoff stress outlined in the notations, we define the *nonsymmetric* nominal stress rate $\overset{\circ}{\boldsymbol{\tau}}$, the classical symmetric Oldroyd stress rate $\overset{\square}{\boldsymbol{\tau}}$ and the *nonsymmetric* convective stress rates $\overset{\nabla}{\boldsymbol{\tau}}$ in the relations

$$\dot{\boldsymbol{\Sigma}}_1^t = \overset{\circ}{\boldsymbol{\tau}}{}^t \cdot \mathbf{F}^{-t} \quad \text{and} \quad \dot{\boldsymbol{\Sigma}}_2^t = \mathbf{F}^{-1} \cdot \overset{\square}{\boldsymbol{\tau}}{}^t \cdot \mathbf{F}^{-t} \quad \text{and} \quad \dot{\mathbf{T}} = \mathbf{F}^t \cdot \overset{\nabla}{\boldsymbol{\tau}}{}^t \cdot \mathbf{F}^{-t}. \tag{1}$$

Thereby, using $\dot{\mathbf{F}} = \mathbf{l} \cdot \mathbf{F}$ and $\dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1} \cdot \mathbf{l}$, the spatial stress rates expand into

$$\overset{\circ}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t \quad \text{and} \quad \overset{\square}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t - \mathbf{l} \cdot \boldsymbol{\tau} \quad \text{and} \quad \overset{\nabla}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t + \mathbf{l}^t \cdot \boldsymbol{\tau}. \tag{2}$$

Equivalently, by considering the relation between the traction vectors $\mathbf{t}_1\,\mathrm{d}A = \mathbf{t}\,\mathrm{d}a$ together with the Nanson formula we obtain the rate format $\dot{\mathbf{t}}_1\,\mathrm{d}A = \overset{\circ}{\mathbf{t}}\,\mathrm{d}a$ with $\dot{\mathbf{t}}_1 = \dot{\boldsymbol{\Sigma}}_1^t \cdot \mathbf{N}$ and $J\overset{\circ}{\mathbf{t}} = \overset{\circ}{\boldsymbol{\tau}}{}^t \cdot \mathbf{n}$. The introduced spatial stress rates are summarized in Table 1.

Next, we introduce the general structure of some useful fourth order tangent stiffness operators for incrementally linear material behaviour. We will not consider incrementally *nonlinear* materials such as the class of *hypoplastic* materials where the stress rates depend nonlinearly on the velocity gradients. The tangent stiffness operators associated with the nominal stress are the fourth order first Lagrange and Euler tangent operators $\mathscr{L}_1$ and $\mathscr{E}_1$, respectively, which are defined through the relations

Table 1. Spatial rates of Kirchhoff stress

$\overset{\square}{\boldsymbol{\tau}}{}^t = \mathscr{E}_2 : \mathbf{l}$ and $\mathbf{l} : \mathscr{E}_2 = \overset{\square}{\boldsymbol{\tau}}{}^t$ with $\overset{\square}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t - \mathbf{l} \cdot \boldsymbol{\tau}$
$\overset{\circ}{\boldsymbol{\tau}}{}^t = \mathscr{E}_1 : \mathbf{l}$ and $\mathbf{l} : \mathscr{E}_1 = \overset{\circ}{\boldsymbol{\tau}}{}^t$ with $\overset{\circ}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t$
$\dot{\boldsymbol{\tau}}^t = \mathscr{E}_0 : \mathbf{l}$ and $\mathbf{l} : \mathscr{E}_\nabla = \dot{\boldsymbol{\tau}}^t$ with $\dot{\boldsymbol{\tau}}^t = \dot{\boldsymbol{\tau}}^t$
$\overset{\nabla}{\boldsymbol{\tau}}{}^t = \mathscr{E}_\nabla : \mathbf{l}$ and $\mathbf{l} : \mathscr{E}_0 = \overset{\nabla}{\boldsymbol{\tau}}{}^t$ with $\overset{\nabla}{\boldsymbol{\tau}}{}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}^t + \mathbf{l}^t \cdot \boldsymbol{\tau}$

Table 2. Euler tangent operators

	$\mathscr{E}_2$	$\mathscr{E}_1$	$\mathscr{E}_0$	$\mathscr{E}_\nabla$
$\mathscr{E}_2$	$\mathscr{E}_2$	$\mathscr{E}_1-\mathbf{I}\overline{\otimes}\boldsymbol{\tau}$	$\mathscr{E}_0-\mathbf{I}\overline{\otimes}\boldsymbol{\tau}-\boldsymbol{\tau}\underline{\otimes}\mathbf{I}$	$\mathscr{E}_\nabla-\mathbf{I}\overline{\otimes}\boldsymbol{\tau}-\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$
$\mathscr{E}_1$	$\mathscr{E}_2+\mathbf{I}\overline{\otimes}\boldsymbol{\tau}$	$\mathscr{E}_1$	$\mathscr{E}_0-\boldsymbol{\tau}\underline{\otimes}\mathbf{I}$	$\mathscr{E}_\nabla-\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$
$\mathscr{E}_0$	$\mathscr{E}_2+\mathbf{I}\overline{\otimes}\boldsymbol{\tau}+\boldsymbol{\tau}\underline{\otimes}\mathbf{I}$	$\mathscr{E}_1+\boldsymbol{\tau}\underline{\otimes}\mathbf{I}$	$\mathscr{E}_0$	$\mathscr{E}_\nabla+\boldsymbol{\tau}\underline{\otimes}\mathbf{I}-\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$
$\mathscr{E}_\nabla$	$\mathscr{E}_2+\mathbf{I}\overline{\otimes}\boldsymbol{\tau}+\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$	$\mathscr{E}_1+\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$	$\mathscr{E}_0-\boldsymbol{\tau}\underline{\otimes}\mathbf{I}+\mathbf{I}\underline{\otimes}\boldsymbol{\tau}$	$\mathscr{E}_\nabla$

$$\dot{\boldsymbol{\Sigma}}_1^t = \mathscr{L}_1 : \dot{\mathbf{F}} \quad \text{and} \quad \overset{\circ}{\boldsymbol{\tau}}{}^t = \mathscr{E}_1 : \mathbf{l} \quad \text{with} \quad \mathscr{E}_1 = [\mathbf{I} \overline{\otimes} \mathbf{F}] : \mathscr{L}_1 : [\mathbf{I} \overline{\otimes} \mathbf{F}^t]. \tag{3}$$

Accordingly, the spatial tangent operators $\mathscr{E}_2$, $\mathscr{E}_0$ and $\mathscr{E}_\nabla$ are introduced as the definitions

$$\overset{\square}{\boldsymbol{\tau}}{}^t = \mathscr{E}_2 : \mathbf{l} \quad \text{and} \quad \dot{\boldsymbol{\tau}}^t = \mathscr{E}_0 : \mathbf{l} \quad \text{and} \quad \overset{\nabla}{\boldsymbol{\tau}}{}^t = \mathscr{E}_\nabla : \mathbf{l} \tag{4}$$

with the relation for example to $\mathscr{E}_1$ easily derived from eqn 2

$$\mathscr{E}_2 = \mathscr{E}_1 - \mathbf{I} \overline{\otimes} \boldsymbol{\tau} \quad \text{and} \quad \mathscr{E}_0 = \mathscr{E}_1 + \boldsymbol{\tau} \underline{\otimes} \mathbf{I} \quad \text{and} \quad \mathscr{E}_\nabla = \mathscr{E}_1 + \mathbf{I} \underline{\otimes} \boldsymbol{\tau}. \tag{5}$$

In order to summarize and interrelate the introduced spatial tangent stiffness operators we refer to Table 2.

Remark

From eqn 5 and the properties of the dyadic product $\underline{\otimes}$, which are pointed out in the notations, we conclude that, for arbitrary second order tensors $\mathbf{a}$, the following identities hold

$$\mathscr{E}_\nabla : \mathbf{a} = \mathbf{a} : \mathscr{E}_0 \quad \text{and} \quad \mathbf{a} : \mathscr{E}_\nabla = \mathscr{E}_0 : \mathbf{a}.$$

Therefore, the operators $\mathscr{E}_\nabla$ and $\mathscr{E}_0$ do *not* possess the property of a *major symmetry*, which would manifest itself in invariance under the left or right action of a tensor $\mathbf{a}$. This is in contrast to the operators $\mathscr{E}_2$ and $\mathscr{E}_1$ as alluded to in Table 1. In the following section, this fundamental observation will play an essential role in the interpretation of the symmetry properties of the *hyperelasto-plastic* tangent operator. □

Remark

For a Cartesian co-ordinate system, the notation used in eqn 3 expands into

$$\left[[\mathbf{I} \overline{\otimes} \mathbf{F}] : \mathscr{L}_1 : [\mathbf{I} \overline{\otimes} \mathbf{F}^t]\right]_{ijkl} \rightarrow F_{jN}[\mathscr{L}_1]_{iNkP} F_{lP}. \qquad \square$$

Remark

The introduced tangent operators and transformations are general for incrementally *linear* material behaviour. Hence, they are in particular valid for *hypoelasticity*, *hyperelasticity*, *hypoelasto-plasticity* (based on the *additive* decomposition of the spatial rate of deformation $\mathbf{l}^{sym}$) as well as for *hyperelasto-plasticity* (based on the *multiplicative* decomposition of $\mathbf{F}$). In the sequel of this work, we will restrict our considerations to the thermodynamically sound *hyperelasto-plastic* constitutive framework. □

3. HYPERELASTO-PLASTIC TANGENT OPERATOR

Within the context of *hyperelasto-plasticity*, the point of departure is the *multiplicative* decomposition and the deformation gradient into an elastic and a plastic part

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p. \tag{6}$$

For *hyperelasticity*, the free energy density Ψ per unit volume of the reference configuration $\mathscr{B}_0$ acts as a potential for the stress response. To this end, Ψ is expressed in terms of $\mathbf{F}_e$ and a set of strainlike internal variables, that control the hardening (or softening). For our purpose, it is sufficient to choose only one scalar internal variable κ, which represents isotropic hardening.

Taking into account the requirements of objectivity under rigid body motions superposed onto the current configuration, we thus assume that

$$\Psi = \hat{\Psi}(\mathbf{F}_e, \kappa) \quad \text{with} \quad \hat{\Psi}(\mathbf{F}_e, \kappa) = \hat{\Psi}(\mathbf{Q} \cdot \mathbf{F}_e, \kappa) \quad \forall \mathbf{Q} \in \mathrm{SO}(3), \tag{7}$$

where SO(3) is the group of proper orthogonal transformations.

The Clausius–Duhem inequality may be expressed in terms of the first Piola–Kirchhoff stress $\bar{\boldsymbol{\Sigma}}_1^t = \boldsymbol{\tau} \cdot \mathbf{F}_e^{-t}$ and $\mathbf{F}_e$, defined on the plastic intermediate configuration $\bar{\mathscr{B}}$, as

$$\boldsymbol{\tau} : \mathbf{l} - \dot{\Psi} = \bar{\boldsymbol{\Sigma}}_1^t : [\mathbf{l} \cdot \mathbf{F}_e] - \dot{\Psi} > 0. \tag{8}$$

Upon introducing the decomposition of the Euler velocity gradient $\mathbf{l}$ into an elastic and a plastic part

$$\mathbf{l} = \mathbf{l}_e + \mathbf{l}_p \quad \text{with} \quad \mathbf{l}_e = \dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1}, \quad \mathbf{l}_p = \mathbf{F}_e \cdot \bar{\mathbf{L}}_p \cdot \mathbf{F}_e^{-1} \quad \text{and} \quad \bar{\mathbf{L}}_p = \dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1} \tag{9}$$

into eqn 8, and taking into account eqn 7, we obtain the *hyperelastic* part of the constitutive law and the remaining dissipation inequality as

$$\bar{\boldsymbol{\Sigma}}_1^t = \partial_{\mathbf{F}_e} \hat{\Psi} \quad \text{and} \quad D = \bar{\mathbf{T}} : \bar{\mathbf{L}}_p + K\dot{\kappa} \geqslant 0. \tag{10}$$

The stress $\bar{\mathbf{T}}$ on $\bar{\mathscr{B}}$, originally introduced by Mandel (1972) and subsequently employed by e.g., Lubliner (1986) and Miehe (1994), and the drag stress K, which is the dissipative stress that is thermodynamically conjugated to κ, are given as

$$\bar{\mathbf{T}} = \mathbf{F}_e^t \cdot \bar{\boldsymbol{\Sigma}}_1^t \quad \text{and} \quad K = -\partial_\kappa \hat{\Psi}. \tag{11}$$

For *non-associated* plasticity the structure of the dissipation inequality suggests a yield condition Φ and a flow rule in terms of the stress measure $\bar{\mathbf{T}}$

$$\Phi = \hat{\Phi}(\bar{\mathbf{T}}, K) \quad \text{with} \quad \partial_{\bar{\mathbf{T}}} \hat{\Phi} = \bar{\mathbf{N}} \quad \text{and} \quad \bar{\mathbf{L}}_p = \lambda \bar{\mathbf{M}} \quad \overset{\text{ass.}}{\rightsquigarrow} \quad \bar{\mathbf{L}}_p = \lambda \bar{\mathbf{N}}. \tag{12}$$

Equivalently, the evolution equation for the hardening variable κ is given in terms of the drag stress

$$\Phi = \hat{\Phi}(\bar{\mathbf{T}}, K) \quad \text{with} \quad \partial_K \hat{\Phi} = \bar{N} \quad \text{and} \quad \dot{\kappa} = \lambda \bar{M} \quad \overset{\text{ass.}}{\rightsquigarrow} \quad \dot{\kappa} = \lambda \bar{N}. \tag{13}$$

Hence, the special case of *associated* plasticity is obtained upon substituting $\bar{\mathbf{M}}$, $\bar{M}$ by $\bar{\mathbf{N}}$, $\bar{N}$, whereby the principle of maximum dissipation will be satisfied, see Mandel (1972) Lubliner (1984, 1986), Miehe and Stein (1992) and Miehe (1994).

Remark

Plastic-loading and *elastic-unloading* conditions together with the requirement of consistency are expressed as

$$\hat{\Phi}(\hat{\mathbf{T}}, K) \leqslant 0 \quad \lambda \geqslant 0 \quad \lambda\hat{\Phi}(\hat{\mathbf{T}}, K) = 0 \quad \text{and} \quad \lambda\dot{\Phi}(\hat{\mathbf{T}}, K) = 0. \tag{14}$$

For *associated* plasticity the loading–unloading conditions follow from the optimality conditions that are implied by the principle of maximum dissipation. In terms of an optimization problem with inequality constraints they represent the classical Kuhn–Tucker complementary conditions. For *non-associated* plasticity the loading–unloading conditions are postulated by merely considering physics. □

Interludium

It proves convenient to introduce the *hyperelastic* operators $\mathscr{E}_0^{el}$, $\mathscr{E}_1^{el}$, $\mathscr{E}_2^{el}$ and $\mathscr{E}_\nabla^{el}$ on $\mathscr{B}$. To demonstrate the similarity in structure of the *hyperelastic* operators with those shown in Table 2, we give below the basic steps leading to these relations. From the transformations between the intermediate and the spatial configuration $\bar{\mathscr{B}}$ and $\mathscr{B}$, i.e., from elastic *pull-back* operations applied to the Kirchhoff stress, we define the *elastic* nominal rate $\overset{\circ}{\boldsymbol{\tau}}^t$ together with $\overset{\square}{\boldsymbol{\tau}}^t$ and $\overset{\nabla}{\boldsymbol{\tau}}$ in the relations

$$\dot{\mathbf{\Sigma}}_1^t = \overset{\circ}{\boldsymbol{\tau}}^t \cdot \mathbf{F}_e^{-t} \quad \text{and} \quad \dot{\mathbf{\Sigma}}_2^t = \mathbf{F}_e^{-1} \cdot \overset{\square}{\boldsymbol{\tau}}^t \cdot \mathbf{F}_e^{-t} \quad \text{and} \quad \dot{\mathbf{T}} = \mathbf{F}_e^t \cdot \overset{\nabla}{\boldsymbol{\tau}}^t \cdot \mathbf{F}_e^{-t}. \tag{15}$$

Thereby, using $\dot{\mathbf{F}}_e = \mathbf{l}_e \cdot \mathbf{F}_e$ and $\dot{\mathbf{F}}_e^{-1} = -\mathbf{F}_e^{-1} \cdot \mathbf{l}_e$, the *elastic* stress rates expand into

$$\overset{\circ}{\boldsymbol{\tau}}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}_e^t \quad \text{and} \quad \overset{\square}{\boldsymbol{\tau}}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}_e^t - \mathbf{l}_e \cdot \boldsymbol{\tau} \quad \text{and} \quad \overset{\nabla}{\boldsymbol{\tau}}^t = \dot{\boldsymbol{\tau}}^t - \boldsymbol{\tau} \cdot \mathbf{l}_e^t + \mathbf{l}_e^t \cdot \boldsymbol{\tau}. \tag{16}$$

The *hyperelastic* tangent operators $\mathscr{L}_1^{el}$ and $\mathscr{E}_1^{el}$ are then defined as

$$\dot{\mathbf{\Sigma}}_1^t = \mathscr{L}_1^{el} : \dot{\mathbf{F}}_e \quad \text{and} \quad \overset{\circ}{\boldsymbol{\tau}}^t = \mathscr{E}_1^{el} : \mathbf{l}_e \tag{17}$$

with the relations for the *hyperelastic* fourth order tangent operators

$$\mathscr{L}_1^{el} = \partial^2_{\mathbf{F}_e \mathbf{F}_e} \hat{\Psi} \quad \text{and} \quad \mathscr{E}_1^{el} = [\mathbf{I} \overline{\otimes} \mathbf{F}_e] : \mathscr{L}_1^{el} : [\mathbf{I} \overline{\otimes} \mathbf{F}_e^t]. \tag{18}$$

Analogously to $\mathscr{E}_0$ and $\mathscr{E}_\nabla$ in eqns 4 and 5, we introduce the *hyperelastic* counterparts via the relations

$$\dot{\boldsymbol{\tau}}^t = \mathscr{E}_0^{el} : \mathbf{l}_e \rightsquigarrow \mathscr{E}_0^{el} = \mathscr{E}_1^{el} + \boldsymbol{\tau} \underline{\otimes} \mathbf{I} \quad \text{and} \quad \overset{\nabla}{\boldsymbol{\tau}}^t = \mathscr{E}_\nabla^{el} : \mathbf{l}_e \rightsquigarrow \mathscr{E}_\nabla^{el} = \mathscr{E}_1^{el} + \mathbf{I} \underline{\otimes} \boldsymbol{\tau}. \tag{19}$$

In conclusion, it appears that the generic Euler tangent operators in Table 2 are valid for *hyperelastic* behaviour if merely $\mathbf{F}_e$ is substituted for $\mathbf{F}$. □

We are now in the position to use the consistency condition in eqn 14_4 in standard fashion

$$\dot{\Phi} = \bar{\mathbf{N}} : \dot{\mathbf{T}} + \bar{N}\dot{K} = \boldsymbol{\nu} : \overset{\nabla}{\boldsymbol{\tau}}^t + \bar{\mathbf{N}}\dot{\mathbf{K}} \quad \text{with} \quad \boldsymbol{\nu} = \mathbf{F}_e \cdot \bar{\mathbf{N}} \cdot \mathbf{F}_e^{-1} \tag{20}$$

where we introduced $\overset{\nabla}{\boldsymbol{\tau}}^t$ from eqn 15_3 and $\boldsymbol{\nu}$ is the spatial equivalent, i.e., the mixed variant *push-forward*, of $\bar{\mathbf{N}}$. Upon introducing the spatial format of the flow rule $\mathbf{l}_p = \lambda\boldsymbol{\mu}$ together with the *hardening modulus H* and considering the evolution equation $\dot{\kappa} = \lambda M$ in $\dot{K} = -H\dot{\kappa}$ and finally inserting these relations together with eqn 19_2 into eqn 20 we obtain

$$\dot{\Phi} = \boldsymbol{\nu} : \mathscr{E}_\nabla^{el} : \mathbf{l} - h\lambda \quad \text{with} \quad h = \bar{N}H\bar{M} + \boldsymbol{\nu} : \mathscr{E}_\nabla^{el} : \boldsymbol{\mu} \quad \text{and} \quad \boldsymbol{\mu} = \mathbf{F}_e \cdot \bar{\mathbf{M}} \cdot \mathbf{F}_e^{-1}. \tag{21}$$

In the case of plastic loading, we thus obtain from eqn 21, equivalently to the small strain case, that $\dot{\Phi} = 0$ renders the plastic multiplier λ the value

$$\lambda = \frac{1}{h}\boldsymbol{v}:\mathcal{E}^{el}_{\nabla}:\mathbf{l} \quad \text{if} \quad \boldsymbol{v}:\mathcal{E}^{el}_{\nabla}:\mathbf{l} > 0. \tag{22}$$

In order to establish the spatial form of the *hyperelasto-plastic* tangent stiffness operator, we firstly consider the relation between the nominal and the elastic nominal rate of the Kirchhoff stress $\overset{\circ}{\boldsymbol{\tau}}{}^t$ and $\hat{\boldsymbol{\tau}}^t$ from eqn 16_1 and secondly incorporate eqn 17_2 into the resulting expression. Consequently, the nominal stress rate in $\mathcal{B}$ follows as

$$\overset{\circ}{\boldsymbol{\tau}}{}^t = \hat{\boldsymbol{\tau}}^t - \boldsymbol{\tau}\cdot\mathbf{l}_p^t \rightarrow \overset{\circ}{\boldsymbol{\tau}}{}^t = \mathcal{E}^{el}_1:\mathbf{l}_e - \boldsymbol{\tau}\cdot\mathbf{l}_p^t \quad \rightsquigarrow \quad \overset{\circ}{\boldsymbol{\tau}}{}^t = \mathcal{E}^{el}_1:\mathbf{l} - \lambda\mathcal{E}^{el}_0:\boldsymbol{\mu}. \tag{23}$$

As a result we obtain the first Euler *hyperelasto-plastic* tangent operator in a remarkably simple format

$$\mathcal{E}^{ep}_1 = \mathcal{E}^{el}_1 - \frac{1}{h}\mathcal{E}^{el}_0:\boldsymbol{\mu}\otimes\boldsymbol{v}:\mathcal{E}^{el}_{\nabla} \quad \text{with} \quad \overset{\circ}{\boldsymbol{\tau}}{}^t = \mathcal{E}^{ep}_1:\mathbf{l}. \tag{24}$$

Remark

The corresponding second Euler tangent operator is *symmetric* for the case of *associated* flow rules, i.e., $\boldsymbol{\mu} = \boldsymbol{v}$ and $\bar{M} = \bar{N}$, since $\boldsymbol{v}:\mathcal{E}^{el}_{\nabla} = \mathcal{E}^{el}_0:\boldsymbol{v}$. In fact, for $H = 0$ our result coincides with the tangent operator for ideal multiplicative *hyperelasto-plasticity* which was first derived, by a somewhat different line of argumentation, in Miehe (1994). Moreover, by considering the relation

$$\boldsymbol{v}:[\mathcal{E}^{el}_{\nabla} - \mathcal{E}^{el}_0] = \boldsymbol{\tau}\cdot\boldsymbol{v}^t - \boldsymbol{v}^t\cdot\boldsymbol{\tau}$$

we conclude that the first Euler tangent operator may be expressed for the case of *associated* flow rules in combination with an *isotropic* free energy function Ψ and an *isotropic* yield condition Φ as

$$\mathcal{E}^{ep}_1 = \mathcal{E}^{el}_1 - \frac{1}{h}\mathcal{E}^{el}_0:\boldsymbol{v}\otimes\boldsymbol{v}:\mathcal{E}^{el}_0 \quad \text{with} \quad h = H\bar{N}^2 + \boldsymbol{v}:\mathcal{E}^{el}_0:\boldsymbol{v}.$$

In this case $\boldsymbol{v}$ and $\boldsymbol{\tau}$ commute and we find that $\boldsymbol{v}:\mathcal{E}^{el}_{\nabla} = \boldsymbol{v}:\mathcal{E}^{el}_0$. □

4. REGULARIZED DISCONTINUITY ACROSS MATERIAL SURFACE

As a preliminary to the subsequent discussion, consider the monotonic function $S(\mathbf{X}):\mathbb{R}^3 \rightarrow \mathbb{R}$ defined on the reference configuration $\mathcal{B}_0$. The material surface Γ_0 attached to $\mathcal{B}_0$ with unit normal $\mathbf{N}$, as shown in Fig. 1, is then defined as

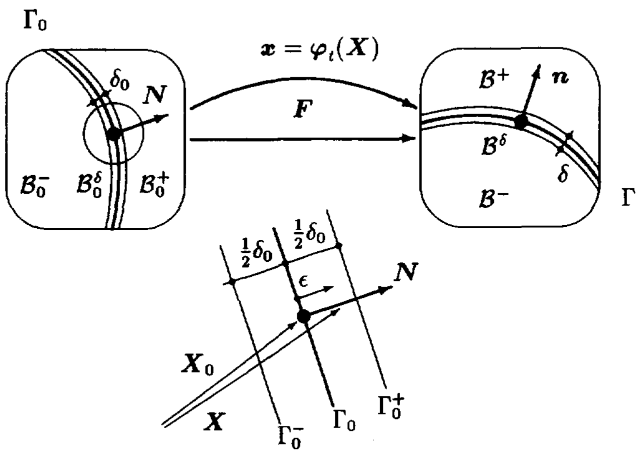

Fig. 1. Regularized discontinuity across material surface.

$$S(\mathbf{X}) = 0 \quad \text{and} \quad \mathbf{N} = j_0^{-1}\nabla_X S \quad \text{with} \quad j_0 = |\nabla_X S|. \tag{25}$$

The surface subdivides $\mathscr{B}_0$ into $\mathscr{B}_0^-$ and $\mathscr{B}_0^+$ with the unit normal $\mathbf{N}$ pointing from $\mathscr{B}_0^-$ to $\mathscr{B}_0^+$. During the deformation $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$ the surface Γ_0 and the unit normal $\mathbf{N}$ are convected to the surface Γ and the unit normal $\mathbf{n}$ in the spatial configuration $\mathscr{B}$ via the relations

$$s(\mathbf{x}, t) = S(\mathbf{X}) \circ \boldsymbol{\varphi}^{-1} = 0 \quad \text{and} \quad \mathbf{n} = j^{-1}\nabla_x s = \frac{j_0}{j}\mathbf{N}\cdot\mathbf{F}^{-1} \quad \text{with} \quad j = |\nabla_x s|. \tag{26}$$

Since the surface Γ_0 is assumed to be attached to $\mathscr{B}_0$, its material time derivative is $\dot{S} = 0$, and the convective velocity c of the spatial surface Γ is given by $c = j^{-1}\partial_t s = -\mathbf{n}\cdot\mathbf{v}$ with $\mathbf{v} = \dot{\boldsymbol{\varphi}}$.

Let us, next, introduce a discontinuous displacement field $\mathbf{u}(\mathbf{X}, t)$ that is spatially smooth except across Γ_0. We may then express this field as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{u}_c(\mathbf{X}, t) + H_S(\mathbf{X})[\![\mathbf{u}(t)]\!], \tag{27}$$

with $\mathbf{u}_c$ the spatially continuous part. Moreover, H_S is the Heaviside function centered on Γ_0, and $[\![\mathbf{u}]\!]$ is the jump of $\mathbf{u}$ across Γ_0. This jump is assumed to be preserved along Γ_0 in the sense that $[\![\mathbf{u}]\!] = \text{const}$ along Γ_0. Thereby, the jump $[\![\mathbf{u}]\!]$ is defined as

$$[\![\mathbf{u}(t)]\!] = \lim_{\varepsilon\to 0}[\mathbf{u}(\mathbf{X}_0 + \varepsilon\mathbf{N}, t) - \mathbf{u}(\mathbf{X}_0 - \varepsilon\mathbf{N}, t)] \quad \text{where} \quad \mathbf{X}_0 \in \Gamma_0. \tag{28}$$

The Heaviside function H_S on Γ_0 is defined via the function $H(\bullet)$ as

$$H_S(\mathbf{X}) = H(S(\mathbf{X})) = \begin{cases} 0 & \text{iff} \quad \mathbf{X} \in \mathscr{B}_0^- \\ 1 & \text{iff} \quad \mathbf{X} \in \Gamma_0 \\ 1 & \text{iff} \quad \mathbf{X} \in \mathscr{B}_0^+ \end{cases} \quad \text{with} \quad H(\bullet) = \begin{cases} 0 & \forall(\bullet) < 0 \\ 1 & \forall(\bullet) \geqslant 0 \end{cases}. \tag{29}$$

To simplify notation we shall in the sequel omit the arguments of all field quantities if there is no risk of confusion. It is noted that the deformation map $\boldsymbol{\varphi}$ is not unique along the discontinuity surface Γ_0, since

$$\boldsymbol{\varphi} = \mathbf{X} + \mathbf{u}_c + H_S[\![\mathbf{u}]\!] = \boldsymbol{\varphi}_c + H_S[\![\mathbf{u}]\!] \quad \text{with} \quad \boldsymbol{\varphi}_c = \mathbf{X} + \mathbf{u}_c, \tag{30}$$

where $\boldsymbol{\varphi}_c$ denotes the continuous part of the total deformation map $\boldsymbol{\varphi}$. The corresponding discontinuous velocity field is

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_c + H_S[\![\dot{\mathbf{u}}]\!]. \tag{31}$$

In order to obtain the deformation gradient corresponding to φ in eqn 30, the scalar-valued Dirac-delta distribution δ_S on Γ_0 is defined via the function $\delta(\bullet)$ as

$$\delta_S = \delta(S(\mathbf{X})) = \begin{cases} 0 & \text{iff} \quad \mathbf{X} \in \mathscr{B}_0^- \\ \infty & \text{iff} \quad \mathbf{X} \in \Gamma_0 \\ 0 & \text{iff} \quad \mathbf{X} \in \mathscr{B}_0^+ \end{cases} \quad \text{with} \quad \delta(\bullet) = \begin{cases} \infty & \forall(\bullet) = 0 \\ 0 & \forall(\bullet) \neq 0 \end{cases}. \tag{32}$$

Moreover, the vector-valued Dirac-delta distribution $\boldsymbol{\delta}_S$ and the Heaviside function H_S on Γ_0 are related in a distributional sense by

$$\int_{\mathscr{B}_0} (\bullet)\cdot\boldsymbol{\delta}_S \,\mathrm{d}V = \int_{\mathscr{B}_0} (\bullet)\cdot\nabla_X H_S \,\mathrm{d}V = \int_{\Gamma_0} (\bullet)\cdot\mathbf{N}\,\mathrm{d}A \quad \forall(\bullet)\in[C_0^\infty(\mathscr{B}_0)]^{ndim(\bullet)}. \tag{33}$$

The last relation follows from integration by parts while taking into account the definition of H_S and $C_0^\infty(\mathscr{B}_0)$. Therefore, the relation between δ_S and H_S on Γ_0 may be stated as the functional identity

$$\nabla_X H_S = \boldsymbol{\delta}_S \quad \text{with} \quad \nabla_X H_S = j_0\,\delta_S \mathbf{N}. \tag{34}$$

Since $[\![\mathbf{u}]\!]$ is spatially constant along Γ_0, we may express the material deformation gradient and its velocity, which are *singular* along the discontinuity surface, as

$$\mathbf{F} = \mathbf{F}_c + j_0\,\delta_S[\![\mathbf{u}]\!]\otimes\mathbf{N} \quad \text{and} \quad \dot{\mathbf{F}} = \dot{\mathbf{F}}_c + j_0\,\delta_S[\![\dot{\mathbf{u}}]\!]\otimes\mathbf{N}. \tag{35}$$

Here, $\mathbf{F}_c = \nabla_X\boldsymbol{\varphi}_c$ and $\dot{\mathbf{F}}_c = \nabla_X\dot{\boldsymbol{\varphi}}_c$ denote the continuous contributions to $\mathbf{F}$ and $\dot{\mathbf{F}}$, respectively.

For a regularization of the singularity terms, we assume that two parallel surfaces Γ_0^- and Γ_0^+ with the same unit normal $\mathbf{N}$ surround a narrow band-shaped domain $\mathscr{B}_0^\delta$ of width δ_0, see Fig. 1. Thereby, δ_0 is much smaller than a typical geometrical length scale of the domain $\mathscr{B}_0$. Hence, we define

$$\mathscr{B}_0^\delta = \left\{\mathbf{X} = \mathbf{X}_0 + \varepsilon\mathbf{N} \,\middle|\, \mathbf{X}_0\in\Gamma_0, \quad -\frac{\delta_0}{2}\leqslant\varepsilon\leqslant\frac{\delta_0}{2}\right\}. \tag{36}$$

We may now proposed the regularized Heaviside function H_R and Dirac-delta distribution δ_R as

$$H_R = \begin{cases} 0 & \text{iff} \quad \mathbf{X}\in\mathscr{B}_0^- \\ \dfrac{1}{2}+\dfrac{\varepsilon}{\delta_0} & \text{iff} \quad \mathbf{X}\in\mathscr{B}_0^\delta \\ 1 & \text{iff} \quad \mathbf{X}\in\mathscr{B}_0^+ \end{cases} \quad \text{and} \quad \delta_R = \begin{cases} \dfrac{1}{j_0\delta_0} & \text{iff} \quad \mathbf{X}\in\mathscr{B}_0^\delta \\ 0 & \text{iff} \quad \mathbf{X}\notin\mathscr{B}_0^\delta \end{cases}. \tag{37}$$

Moreover, the regularized vector-valued Dirac-delta distribution $\boldsymbol{\delta}_R$ is defined as

$$\boldsymbol{\delta}_R = j_0\,\delta_R\mathbf{N} = \begin{cases} \dfrac{1}{\delta_0}\mathbf{N} & \text{iff} \quad \mathbf{X}\in\mathscr{B}_0^\delta \\ 0 & \text{iff} \quad \mathbf{X}\notin\mathscr{B}_0^\delta \end{cases}. \tag{38}$$

Thereby, the regularized vector valued Dirac-delta distribution satisfies the following relation

$$\int_{\mathscr{B}_0} (\bullet)\cdot\boldsymbol{\delta}_R\,\mathrm{d}V = \frac{1}{\delta_0}\int_{\mathscr{B}_0^\delta} (\bullet)\cdot\mathbf{N}\,\mathrm{d}V \rightarrow \int_{\Gamma_0} (\bullet)\cdot\mathbf{N}\,\mathrm{d}A \quad \text{for} \quad \delta_0\rightarrow 0. \tag{39}$$

This result holds also for a finite band width δ_0 if we assume that the quantity $(\bullet)$ does not vary across the band.

The regularized C^0-continuous deformation $\boldsymbol{\varphi}$ together with the regularized displacement and velocity fields are then expressed in terms of the regularized Heaviside function H_R as

$$\boldsymbol{\varphi}_r = \boldsymbol{\varphi}_c + H_R[\![\mathbf{u}]\!] \quad \text{and} \quad \mathbf{u}_r = \mathbf{u}_c + H_R[\![\mathbf{u}]\!] \quad \text{and} \quad \dot{\mathbf{u}}_r = \dot{\mathbf{u}}_c + H_R[\![\dot{\mathbf{u}}]\!]. \tag{40}$$

It follows immediately that the regularized versions of $\mathbf{F}$ and $\dot{\mathbf{F}}$ can be expressed as in eqn 35 by simply replacing δ_S with δ_R

$$\mathbf{F} \to \mathbf{F}_c + j_0\,\delta_R[\![\mathbf{u}]\!] \otimes \mathbf{N} \quad \text{and} \quad \dot{\mathbf{F}} \to \dot{\mathbf{F}}_c + j_0\,\delta_R[\![\dot{\mathbf{u}}]\!] \otimes \mathbf{N}. \tag{41}$$

Subsequently, we consider the restriction to the band-shaped domain $\mathscr{B}_0^\delta$ to obtain the regularized $\mathbf{F}_r$ and $\dot{\mathbf{F}}_r$, which are *discontinuous*, as

$$\mathbf{F}_r = \mathbf{F}_c + \frac{1}{\delta_0}[\![\mathbf{u}]\!] \otimes \mathbf{N} \quad \text{and} \quad \dot{\mathbf{F}}_r = \dot{\mathbf{F}}_c + \frac{1}{\delta_0}[\![\dot{\mathbf{u}}]\!] \otimes \mathbf{N}. \tag{42}$$

As the spatial counterpart of $\dot{\mathbf{F}}_r$, we now consider the *regularized spatial velocity gradient*, $\mathbf{l}_r$ defined by

$$\mathbf{l}_r = \dot{\mathbf{F}}_r \cdot \mathbf{F}_r^{-1}. \tag{43}$$

In order to obtain the structure of $\mathbf{l}_r$ we first evaluate $\mathbf{F}_r^{-1}$ by direct application of the Sherman–Morrison formula, which expresses the inverse of a regular second order tensor with rank-one update, as

$$\mathbf{F}_r^{-1} = \mathbf{F}_c^{-1} - \frac{[\![\mathbf{U}]\!] \otimes \mathbf{N} \cdot \mathbf{F}_c^{-1}}{\delta_0 + \mathbf{N} \cdot [\![\mathbf{U}]\!]} \quad \text{with} \quad [\![\mathbf{U}]\!] = \mathbf{F}_c^{-1} \cdot [\![\mathbf{u}]\!]. \tag{44}$$

Here we introduced the *contravariant pull-back* $[\![\mathbf{U}]\!]$ of the vector $[\![\mathbf{u}]\!]$ with the continuous portion of the regularized deformation gradient. Upon introducing the spatial thickness δ of the localization band via the relation $j\delta = j_0\,\delta_0$, we may also introduce the spatial normal to the band via the relation

$$\mathbf{n} = \frac{j_0}{j}\mathbf{N} \cdot \mathbf{F}_r^{-1} = \delta \frac{\mathbf{N} \cdot \mathbf{F}_c^{-1}}{\delta_0 + \mathbf{N} \cdot [\![\mathbf{U}]\!]}. \tag{45}$$

We thus obtain the interesting representation for $\mathbf{F}_r^{-1}$ in comparison to $\mathbf{F}_r$ as

$$\mathbf{F}_r^{-1} = \mathbf{F}_c^{-1} - \frac{1}{\delta}[\![\mathbf{U}]\!] \otimes \mathbf{n} \quad \text{vs} \quad \mathbf{F}_r = \mathbf{F}_c + \frac{1}{\delta_0}[\![\mathbf{u}]\!] \otimes \mathbf{N}. \tag{46}$$

The regularized spatial velocity gradient $\mathbf{l}_r$ can now be derived in a straightforward fashion and we obtain the representation of $\mathbf{l}_r$ in comparison to the material counterpart $\dot{\mathbf{F}}_r$ as

$$\mathbf{l}_r = \mathbf{l}_c + \frac{1}{\delta}[\![\overset{\circ}{\mathbf{u}}]\!] \otimes \mathbf{n} \quad \text{vs} \quad \dot{\mathbf{F}}_r = \dot{\mathbf{F}}_c + \frac{1}{\delta_0}[\![\dot{\mathbf{u}}]\!] \otimes \mathbf{N}. \tag{47}$$

It appears that the introduced rate $[\![\overset{\circ}{\mathbf{u}}]\!]$ has the structure of a Lie derivative of $[\![\mathbf{u}]\!]$ with respect to the continuous part of the deformation gradient $\mathbf{F}_c$, since we have

$$[\![\mathbf{U}]\!] = \mathbf{F}_c^{-1} \cdot [\![\mathbf{u}]\!] \quad \rightsquigarrow \quad [\![\dot{\mathbf{U}}]\!] = \mathbf{F}_c^{-1} \cdot [\![\overset{\circ}{\mathbf{u}}]\!] \quad \text{with} \quad [\![\overset{\circ}{\mathbf{u}}]\!] = [\![\dot{\mathbf{u}}]\!] - \mathbf{l}_c \cdot [\![\mathbf{u}]\!]. \tag{48}$$

Remark

The classical localization analysis, see, e.g., Rice (1976), assumes *continuity* of the deformation $\boldsymbol{\varphi}$ and its associated deformation gradient $\mathbf{F} = \nabla_X \boldsymbol{\varphi}$. Then the velocity field may be *discontinuous* across Γ_0 with *singular* material gradient. Accordingly, $\dot{\mathbf{u}}_r$ and $\dot{\mathbf{F}}_r$ take on the same structure as in eqns 40 and 42. However, due to the *continuity* of $\mathbf{F}$ it follows that $\mathbf{l}_r$ will be simplified, as compared to eqn 47, in the following fashion

$$\mathbf{l}_r = \dot{\mathbf{F}}_r \cdot \mathbf{F}^{-1} \quad \text{with} \quad \mathbf{n} = \frac{j_0}{j} \mathbf{N} \cdot \mathbf{F}^{-1} \quad \leadsto \quad \mathbf{l}_r = \mathbf{l}_c + \frac{1}{\delta} [\![\dot{\mathbf{u}}]\!] \otimes \mathbf{n}.$$

In the more general approach considered in this paper, *onset* of localization is defined by $[\![\mathbf{u}]\!] = \mathbf{o}$, i.e., $\mathbf{F}_r = \mathbf{F}_c = \mathbf{F}$, whereas $[\![\dot{\mathbf{u}}]\!] \neq \mathbf{o}$. This gives $[\![\overset{\circ}{\mathbf{u}}]\!] = [\![\dot{\mathbf{u}}]\!]$, and the classical approach coincides with the general one at this particular moment.

In the sequel we shall employ the generic expressions for the regularized material and spatial velocity gradients

$$\dot{\mathbf{F}}_r = \dot{\mathbf{F}}_c + \frac{1}{\delta_0} \mathbf{M} \otimes \mathbf{N} \quad \text{and} \quad \mathbf{l}_r = \mathbf{l}_c + \frac{1}{\delta} \mathbf{m} \otimes \mathbf{n}$$

with $\mathbf{M} = [\![\dot{\mathbf{u}}]\!]$ and $\mathbf{m} = [\![\overset{\circ}{\mathbf{u}}]\!]$ or $\mathbf{m} = [\![\dot{\mathbf{u}}]\!]$. □

5. LOCALIZATION: GENERAL APPROACH

The key requirement for the possible development of a regularized strong discontinuity is that the incremental equilibrium equation is spatially continuous across the band. This implies that the traction vector is continuous across the band, which may be expressed in a spatial format via the nominal traction rate as

$$\overset{\circ}{\mathbf{t}}(\mathbf{x}_-) = \overset{\circ}{\mathbf{t}}(\mathbf{x}_0) = \overset{\circ}{\mathbf{t}}(\mathbf{x}_+). \tag{49}$$

Here, we introduced the notation $\mathbf{x}_- = \boldsymbol{\varphi}(\mathbf{X}_-) \in \mathscr{B}^-$ and $\mathbf{x}_+ = \boldsymbol{\varphi}(\mathbf{X}_+) \in \mathscr{B}^+$ such that

$$\mathbf{x}_- = \mathbf{x}_0 - \frac{\delta}{2}\mathbf{n} \quad \text{and} \quad \mathbf{x}_+ = \mathbf{x}_0 + \frac{\delta}{2}\mathbf{n} \quad \text{with} \quad \mathbf{x}_0 = \boldsymbol{\varphi}(\mathbf{X}_0) \in \Gamma. \tag{50}$$

We also note that, in addition to equilibrium continuity, $\mathbf{F}_c$ and $\mathbf{l}_c$ coincide on both sides of the band for δ being a small measure, i.e.,

$$\mathbf{F}_c(\mathbf{X}_-) = \mathbf{F}_c(\mathbf{X}_0) = \mathbf{F}_c(\mathbf{X}_+) \quad \text{and} \quad \mathbf{l}_c(\mathbf{x}_-) = \mathbf{l}_c(\mathbf{x}_0) = \mathbf{l}_c(\mathbf{x}_+). \tag{51}$$

Next, we use the definition of the nominal traction rate

$$\overset{\circ}{\mathbf{t}} = J^{-1} \overset{\circ}{\boldsymbol{\tau}}{}^t \cdot \mathbf{n} = [J^{-1} \mathscr{E}_1 : \mathbf{l}] \cdot \mathbf{n} \tag{52}$$

and insert $\mathbf{l}_c$ and $\mathbf{l}_r$ to evaluate $\overset{\circ}{\mathbf{t}}_b(\mathbf{x}_0)$ and $\overset{\circ}{\mathbf{t}}_\pm = \overset{\circ}{\mathbf{t}}(\mathbf{x}_\pm)$ inside and outside the band

$$\overset{\circ}{\mathbf{t}}_\pm = [J_c^{-1} \mathscr{E}_1(\mathbf{F}_c) : \mathbf{l}_c] \cdot \mathbf{n} \quad \text{and} \quad \overset{\circ}{\mathbf{t}}_b = \overset{\circ}{\mathbf{t}}_\pm + \big[[\![J^{-1} \mathscr{E}_1]\!] : \mathbf{l}_c\big] \cdot \mathbf{n} + \frac{1}{\delta} J_r^{-1} \mathbf{q} \cdot \mathbf{m}. \tag{53}$$

Here, we introduced the *spatial localization tensor* $\mathbf{q} = \mathbf{q}(\mathbf{F}_r)$ as the contraction of the first Euler tangent operator with the surface unit normal $\mathbf{n}$ to obtain

$$\mathbf{q} = \mathbf{n} \cdot \mathscr{E}_1 \cdot \mathbf{n} = \mathbf{n} \cdot \mathscr{E}_2 \cdot \mathbf{n} + \tau_n \mathbf{I} \tag{54}$$

with $\mathscr{E}_{1,2} = \mathscr{E}_{1,2}(\mathbf{F}_r)$ and $\tau_n = \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n}$. Thereby, the contractions are performed with respect to the second and fourth index of the fourth order tensors $\mathscr{E}_{1,2}$. The localization tensor $\mathbf{q}$ will take on the values $\mathbf{q}_{el}$ or $\mathbf{q}_{ep}$ depending on whether elastic or plastic loading takes place inside the band. Moreover, we defined the jump in the tangent stiffness

$$[\![J^{-1}\mathscr{E}_1]\!] = J_r^{-1}\mathscr{E}_1(\mathbf{F}_r) - J_c^{-1}\mathscr{E}_1(\mathbf{F}_c) \tag{55}$$

since $\mathscr{E}_1(\mathbf{F}_r)$ and $\mathscr{E}_1(\mathbf{F}_c)$ take, in general, different values in $\mathscr{B}^{\pm}$ and $\mathscr{B}^{\delta}$ due to the difference of $\mathbf{F}_r$ and $\mathbf{F}_c$.

From the traction continuity in eqn 49 we are now in the position to establish the *localization* condition, or rather the *admissibility* condition for maintaining a regularized strong discontinuity, in the spatial setting as follows:

$$\left[[\![J^{-1}\mathscr{E}_1]\!]:\mathbf{l}_c\right]\cdot\mathbf{n} + \frac{1}{\delta}J_r^{-1}\mathbf{q}\cdot\mathbf{m} = \mathbf{o}. \tag{56}$$

Remark

The localization tensor $\mathbf{q}$ is naturally decomposed into a material and a geometrical contribution, which stems from the structure of the tangent operator $\mathscr{E}_1$. It is quite remarkable that the geometrical part is expressed solely in terms of the normal stress $\tau_n = \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n}$ acting on the discontinuity surface. This fact has not been pointed out in the literature so far. □

6. LOCALIZATION: HYPERELASTICITY IN $\mathscr{B}^{\delta}$ AND $\mathscr{B}^{-}$

For *hyperelasticity*, the only situation of interest is that the displacements are continuous across the boundaries of an *anticipated* band at the current state of deformation, i.e., *no* band has developed so far. Then the hyperelastic tangent operator will take the same value *inside* and *outside* the band

$$J_c^{-1}\mathscr{E}_1^{el}(\mathbf{F}_c) = J_r^{-1}\mathscr{E}_1^{el}(\mathbf{F}_r) \quad \rightsquigarrow \quad [\![J^{-1}\mathscr{E}_1]\!] = \mathbf{o}. \tag{57}$$

Upon introducing this result into in eqn 56, we obtain the condition for *onset* of a regularized displacement discontinuity in a purely *hyperelastic* material as follows

$$\mathbf{q}_{el}\cdot\mathbf{m} = \mathbf{o} \quad \rightsquigarrow \quad \mathbf{m} \neq \mathbf{o} \quad \text{if} \quad \det\mathbf{q}_{el} = 0 \tag{58}$$

where $\mathbf{q}_{el}$ is the *hyperelastic* localization tensor. It may be noted that the width δ of the band does not come into play. Moreover, according to eqn 54, $\mathbf{q}_{el}$ is *symmetric* since the corresponding tangent operator $\mathscr{E}_1^{el}$ possesses major symmetries.

Remark

It is recalled that the Legendre–Hadamard condition, or the condition for *strong ellipticity*, is

$$[\boldsymbol{\alpha}\otimes\mathbf{B}]:\mathscr{L}_1^{el}:[\boldsymbol{\alpha}\otimes\mathbf{B}] > 0 \quad \text{and} \quad [\boldsymbol{\alpha}\otimes\boldsymbol{\beta}]:\mathscr{E}_1^{el}:[\boldsymbol{\alpha}\otimes\boldsymbol{\beta}] > 0$$

for all vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\mathbf{B}$, such that

$$[\boldsymbol{\alpha}\otimes\mathbf{B}] \neq \mathbf{o} \quad \text{and} \quad [\boldsymbol{\alpha}\otimes\boldsymbol{\beta}] \neq \mathbf{o} \quad \text{with} \quad \mathbf{B} = \boldsymbol{\beta}\cdot\mathbf{F}.$$

This is a sufficient requirement for *hyperelastic* materials to assure uniqueness of solutions to boundary value problems, see Ogden (1984). Clearly, by identifying the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with the vectors $\mathbf{m}$ and $\mathbf{n}$, the condition for *strong ellipticity* is recovered in terms of the *hyperelastic* localization tensor as $\mathbf{m}\cdot\mathbf{q}_{el}\cdot\mathbf{m} > 0$. Therefore, the possibility of a bifurcation into a band-shaped mode is excluded for *hyperelastic* materials as long as the *strong ellipticity* requirement is satisfied. □

7. LOCALIZATION: HYPERELASTO-PLASTICITY IN $\mathcal{B}^\delta$ AND $\mathcal{B}^-$

For *hyperelasto-plasticity*, the situation of primary interest is, again, that displacements are continuous across the boundaries of an *anticipated* band at the current state of deformation, i.e., *no* band has developed so far. We first consider the case of *plastic loading* inside *and* outside the band, whereby the *hyperelasto-plastic* tangent operator takes the same value *inside* and *outside* the band

$$J_c^{-1}\mathcal{E}_1^{ep}(\mathbf{F}_c) = J_r^{-1}\mathcal{E}_1^{ep}(\mathbf{F}_r) \quad \rightsquigarrow \quad [\![J^{-1}\mathcal{E}_1]\!] = \mathbf{o}. \tag{59}$$

Upon introducing this result into eqn 56, we obtain the condition for *onset* of a regularized displacement discontinuity in a *hyperelasto-plastic* material as follows:

$$\mathbf{q}_{ep}\cdot\mathbf{m} = \mathbf{o} \quad \rightsquigarrow \quad \mathbf{m} \neq \mathbf{o} \quad \text{if} \quad \det\mathbf{q}_{ep} = 0 \tag{60}$$

where $\mathbf{q}_{ep}$ is the *hyperelasto-plastic* localization tensor. Again, the width δ of the band does not come into play. Moreover, according to eqn 54, $\mathbf{q}_{ep}$ is *symmetric* as long as the corresponding tangent operator $\mathcal{E}_1^{ep}$ possesses major symmetries.

Taking into account the structure of $\mathcal{E}_1^{ep}$ in eqn 24, a general representation of the corresponding $\mathbf{q}_{ep}$ is

$$\mathbf{q}_{ep} = \mathbf{q}_{el} - \frac{1}{h}\mathbf{e}_\mu \otimes \mathbf{e}_\nu \quad \text{with} \quad \mathbf{e}_\mu = [\mathcal{E}_0^{el} : \boldsymbol{\mu}]\cdot\mathbf{n} \quad \text{and} \quad \mathbf{e}_\nu = [\boldsymbol{\nu} : \mathcal{E}_\nabla^{el}]\cdot\mathbf{n}. \tag{61}$$

The explicit expression for the critical hardening modulus, for which $\mathbf{q}_{ep}$ first becomes singular, can be determined in a fashion that is quite similar to that of the geometrically linear theory, see, e.g., Ottosen and Runesson (1991). To this end, we use the following auxiliary result for the determinant of a second order tensor $\mathbf{A}$, that is defined as the sum of a regular second order tensor $\mathbf{B}$ and a rank-one update

$$\mathbf{A} = \mathbf{B} + \alpha\mathbf{c}\otimes\mathbf{d} \quad \rightsquigarrow \quad \det\mathbf{A} = \det\mathbf{B}[1 + \alpha\mathbf{d}\cdot\mathbf{B}^{-1}\cdot\mathbf{c}].$$

The proof follows by using the definition of the determinant of a second order tensor in terms of its invariants and exploiting the Cayley–Hamilton theorem. Hence, we obtain

$$\det\mathbf{q}_{ep} = \det\mathbf{q}_{el}\left[1 - \frac{1}{h}\mathbf{e}_\nu\cdot\mathbf{q}_{el}^{-1}\cdot\mathbf{e}_\mu\right] \tag{62}$$

from which we derive the critical hardening modulus H_{cr}, rendering $\det\mathbf{q}_{ep} = 0$, as

$$\bar{N}H_{cr}\bar{M} = \max_{|\mathbf{n}|=1}(\mathbf{e}_\nu\cdot\mathbf{q}_{el}^{-1}\cdot\mathbf{e}_\mu) - \boldsymbol{\nu} : \mathcal{E}_\nabla^{el} : \boldsymbol{\mu}. \tag{63}$$

We conclude that no localization can occur as long as $H > H_{cr}$.

Remark

It is interesting to interpret the preceding results in terms of the solution of the following general *right eigenvalue problem* incorporating the *hyperelastic* localization tensor as a metric

$$\mathbf{q}_{ep}\cdot\mathbf{z} = \omega\mathbf{q}_{el}\cdot\mathbf{z}. \tag{64}$$

The corresponding standard right eigenvalue problem is given by

$$\mathbf{q}_{el}^{-1} \cdot \mathbf{q}_{ep} \cdot \mathbf{z} = \omega \mathbf{z} \quad \rightsquigarrow \quad \left[[1-\omega]\mathbf{I} - \frac{1}{h} \mathbf{q}_{el}^{-1} \cdot [\mathbf{e}_{\mu} \otimes \mathbf{e}_{\nu}] \right] \cdot \mathbf{z} = \mathbf{o}. \tag{65}$$

With the lemma given above the characteristic equation follows straightforward

$$[1-\omega]^2 \left[[1-\omega] - \frac{1}{h} \mathbf{e}_{\nu} \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_{\mu} \right] = 0 \tag{66}$$

with solution for the eigenvalues

$$\omega_1 = 1, \quad \omega_2 = 1 \quad \text{and} \quad \omega_3 = 1 - \frac{1}{h} \mathbf{e}_{\nu} \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_{\mu}. \tag{67}$$

It is easily verified that the corresponding right eigenmodes follow from

$$\mathbf{e}_{\nu} \cdot \mathbf{z}_1 = 0, \quad \mathbf{e}_{\nu} \cdot \mathbf{z}_2 = 0 \quad \text{and} \quad \mathbf{z}_3 = \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_{\mu}. \tag{68}$$

Observe the result for the determinant of the localization tensor $\det \mathbf{q}_{ep} = \omega_3 \det \mathbf{q}_{el}$. □

8. LOCALIZATION: HYPERELASTO-PLASTICITY IN $\mathscr{B}^{\delta}$

Another possible loading situation within *hyperelasto-plasticity* is characterized by *plastic loading* inside and *elastic loading* outside the *anticipated* band, while it is still assumed that *no* band has developed so far. The tangent operator within the band will then get an additional contribution due to the *plastic loading* condition in $\mathscr{B}^{\delta}$ such that

$$J_c^{-1} \mathscr{E}_1^{el}(\mathbf{F}_c) \neq J_r^{-1} \mathscr{E}_1^{ep}(\mathbf{F}_r) \quad \text{and} \quad [\![J^{-1} \mathscr{E}_1]\!] = -J^{-1} \frac{1}{h} \mathscr{E}_0^{el} : \boldsymbol{\mu} \otimes \boldsymbol{\nu} : \mathscr{E}_{\nabla}^{el}. \tag{69}$$

Upon introducing this result into eqn 56 we obtain the condition for *onset* of a regularized displacement discontinuity as follows:

$$\frac{1}{\delta} \mathbf{q}_{ep} \cdot \mathbf{m} = \lambda_c \mathbf{e}_{\mu}. \tag{70}$$

Here, λ_c denotes the *negative* 'plastic multiplier', which reflects the *elastic unloading* condition outside the band

$$\lambda_c = \frac{1}{h} \boldsymbol{\nu} : \mathscr{E}_{\nabla}^{el} : \mathbf{l}_c < 0. \tag{71}$$

With the inverse of the elastoplastic localization tensor $\mathbf{q}_{ep}$ given as

$$\mathbf{q}_{ep}^{-1} = \mathbf{q}_{el}^{-1} + \frac{\mathbf{q}_{el}^{-1} \cdot \mathbf{e}_{\mu} \otimes \mathbf{e}_{\nu} \cdot \mathbf{q}_{el}^{-1}}{h - \mathbf{e}_{\nu} \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_{\mu}} \tag{72}$$

we may solve for $\mathbf{m}$ in terms of the eigensolution $[\omega_3, \mathbf{z}_3]$ of $\mathbf{q}_{ep}$

$$\frac{1}{\delta}\mathbf{m} = \gamma \mathbf{z}_3 \quad \text{with} \quad \gamma = \frac{\mathbf{v} : \mathscr{E}_\nabla^{el} : \mathbf{l}_c}{h - \mathbf{e}_v \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_\mu} = \frac{\lambda_c}{\omega_3} > 0 \quad \forall \omega_3 < 0. \tag{73}$$

Remark

For the present case, the localization condition contains the width δ of the band. Moreover, the magnitude γ of the spatial jump $\mathbf{m}$ is driven by the continuous part $\mathbf{l}_c$ of the spatial velocity gradient. □

9. LOCALIZATION ANALYSIS FOR ISOTROPIC MODELS

In this section we elaborate on the expression in eqn 63 in order to establish more explicit representations that can be used towards an analytical evaluation of critical directions $\mathbf{n}_{cr}$ and the corresponding critical hardening modulus H_{cr}. For the sake of simplicity, we make the significant restriction to *isotropic hyperelasto-plastic* materials. Then the flow direction $\boldsymbol{\mu}$, the normal to the yield surface $\mathbf{v}$, the elastic Finger tensor $\mathbf{b}_e$ and the Kirchhoff stress $\boldsymbol{\tau}$ are coaxial, i.e., they commute.

First, as a generic model of *hyperelasticity*, we choose the *isotropic* compressible Neo–Hooke material for which the free energy is formulated in terms of the first invariant $I_1 = \mathbf{F}_e : \mathbf{F}_e$, the Jacobi determinant $J_e = \det \mathbf{F}_e$ and two material parameters G and L as

$$\Psi = \tfrac{1}{2} G[I_1 - 3] + U(J_e) \quad \text{with} \quad U(J_e) = \tfrac{1}{2} L \ln^2 J_e - G \ln J_e. \tag{74}$$

Then, the spatial Kirchhoff stress tensor and the second Euler *hyperelastic* tangent operator follow as

$$\boldsymbol{\tau} = G[\mathbf{b}_e - \mathbf{I}] + L \ln J_e \mathbf{I} \quad \text{and} \quad \mathscr{E}_2^{el} = L\mathbf{I} \otimes \mathbf{I} + 2[G - L \ln J_e]\mathscr{I} \tag{75}$$

with $\mathscr{I} = 1/2[\mathbf{I} \overline{\otimes} \mathbf{I} + \mathbf{I} \underline{\otimes} \mathbf{I}]$ denoting the *symmetric* fourth order unit tensor. By noting that $\mathbf{n} \cdot \mathscr{I} \cdot \mathbf{n} = 1/2[\mathbf{I} + \mathbf{n} \otimes \mathbf{n}]$ and with $b_n^e = \mathbf{n} \cdot \mathbf{b}_e \cdot \mathbf{n}$ we then compute

$$\mathbf{n} \cdot \mathscr{E}_2^{el} \cdot \mathbf{n} = \big[G + L[1 - \ln J_e]\big]\mathbf{n} \otimes \mathbf{n} + [G - L \ln J_e]\mathbf{I} \quad \text{and} \quad \tau_n = G b_n^e - [G - L \ln J_e]. \tag{76}$$

Hence, we obtain the *hyperelastic* spatial localization tensor $\mathbf{q}_{el}$ from eqn 54 as

$$\mathbf{q}_{el} = G b_n^e \mathbf{I} + \big[G + L[1 - \ln J_e]\big]\mathbf{n} \otimes \mathbf{n}. \tag{77}$$

For later use we compute the inverse of $\mathbf{q}_{el}$ in closed form by the Sherman–Morrison formula as

$$\mathbf{q}_{el}^{-1} = \frac{1}{G b_n^e}[\mathbf{I} - \alpha \mathbf{n} \otimes \mathbf{n}] \quad \text{with} \quad \alpha = \frac{G + L[1 - \ln J_e]}{G[b_n^e + 1] + L[1 - \ln J_e]}. \tag{78}$$

Next we observe that, from the definitions of $\mathscr{E}_0^{el}$ and $\mathscr{E}_\nabla^{el}$ in Table 2 in terms of $\mathscr{E}_2^{el}$, and due to the assumption of *isotropy*, the following results hold:

$$\mathscr{E}_0^{el} : \boldsymbol{\mu} = \mathscr{E}_2^{el} : \boldsymbol{\mu} + 2\boldsymbol{\tau} \cdot \boldsymbol{\mu} \quad \text{and} \quad \mathbf{v} : \mathscr{E}_\nabla^{el} = \mathbf{v} : \mathscr{E}_2^{el} + 2\mathbf{v} \cdot \boldsymbol{\tau}. \tag{79}$$

Incorporating the expressions for $\mathscr{E}_2^{el}$ and $\boldsymbol{\tau}$ in eqn 75 and introducing the notation $\mu_1 = \boldsymbol{\mu} : \mathbf{I}$ and $v_1 = \mathbf{v} : \mathbf{I}$ for the first invariants of $\boldsymbol{\mu}$ and $\mathbf{v}$, we obtain the key representations for the subsequent derivations

$$\mathscr{E}_0^{el} : \boldsymbol{\mu} = 2G\mathbf{b}_e \cdot \boldsymbol{\mu} + L\mu_1 \mathbf{I} \quad \text{and} \quad \mathbf{v} : \mathscr{E}_V^{el} = 2G\mathbf{v} \cdot \mathbf{b}_e + L\nu_1 \mathbf{I} \tag{80}$$

together with the quadratic form

$$\mathbf{v} : \mathscr{E}_V^{el} : \boldsymbol{\mu} = \mathbf{v} : \mathscr{E}_0^{el} : \boldsymbol{\mu} = 2G[\mathbf{v} \cdot \mathbf{b}_e] : \boldsymbol{\mu} + L\nu_1 \mu_1 . \tag{81}$$

Moreover, we obtain the following expressions for the vectors $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$

$$\mathbf{e}_\mu = 2G\mathbf{b}_e \cdot \boldsymbol{\mu} \cdot \mathbf{n} + L\mu_1 \mathbf{n} \quad \text{and} \quad \mathbf{e}_\nu = 2G\mathbf{b}_e \cdot \mathbf{v} \cdot \mathbf{n} + L\nu_1 \mathbf{n}. \tag{82}$$

We are now in the position to compute the critical hardening modulus H_{cr} from the optimization problem

$$\frac{\bar{N}\bar{M}}{2G} H_{cr} = \max_{|\mathbf{n}|=1} Y(\mathbf{n}) - [\mathbf{v} \cdot \mathbf{b}_e] : \boldsymbol{\mu} - \frac{L}{2G} \nu_1 \mu_1 . \tag{83}$$

To obtain this expression, we incorporated the inverse of the *hyperelastic* spatial localization tensor in eqn 78 into the quadratic form $\mathbf{e}_\nu \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_\mu$ to render the nondimensional function

$$\begin{aligned} Y(\mathbf{n}) &= \frac{1}{2G} \mathbf{e}_\nu \cdot \mathbf{q}_{el}^{-1} \cdot \mathbf{e}_\mu \\ &= \frac{1}{b_n^e} \Bigg[2\mathbf{n} \cdot \mathbf{v} \cdot \mathbf{b}_e^2 \cdot \boldsymbol{\mu} \cdot \mathbf{n} + [1-\alpha] \frac{L}{G} \mathbf{n} \cdot \mathbf{b}_e \cdot [\mu_1 \mathbf{v} + \nu_1 \boldsymbol{\mu}] \cdot \mathbf{n} \\ &\quad - 2\alpha [\mathbf{n} \cdot \mathbf{b}_e \cdot \mathbf{v} \cdot \mathbf{n}][\mathbf{n} \cdot \mathbf{b}_e \cdot \boldsymbol{\mu} \cdot \mathbf{n}] + [1-\alpha] \frac{L^2}{2G^2} \nu_1 \mu_1 \Bigg]. \end{aligned} \tag{84}$$

Then the critical direction $\mathbf{n}_{cr}$ is given by the solution

$$\mathbf{n}_{cr} = \arg\left(\max_{|\mathbf{n}|=1} Y(\mathbf{n}) \right). \tag{85}$$

It is remarkable that we retrieve the representation of the geometrically linear theory for the assumption of *small* elastic strains, i.e., at the identity $\mathbf{F}_e = \mathbf{I} \rightsquigarrow \mathbf{b}_e = \mathbf{I}$ and $b_n^e = 1$. Thereby, the amount of plastic straining does not appear *explicitly* in eqn 84 but is only involved via the difference between the total and the elastic deformation gradient $\mathbf{F}_p = \mathbf{F}_e^{-1} \cdot \mathbf{F}$. On the other hand, in comparing the critical hardening modulus H_{cr} from eqn 83 with the actual hardening modulus $H(\kappa)$, the amount of plastic strain will play an essential role since it indirectly defines the hardening variable κ. Due to the *hyperelastic* part of the constitutive framework the elastic part $\mathbf{b}_e$ of the total *elasto-plastic* straining is directly responsible for the stress state which in turn determines the directions $\mathbf{v}$ and $\boldsymbol{\mu}$. Thus, either the stress state or the elastic strains could be told to be responsible for the solution $\mathbf{n}_{cr}$ in eqn 85. Thereby, the amount of the elastic part of the total *elasto-plastic* straining is contained in the elastic Finger tensor which acts as a metric within the representation of the function $Y(\mathbf{n})$ in eqn 84. Analytical solutions with respect to the principal axes of general stress states in the case of the geometrically linear theory were derived in Ottosen and Runesson (1991), to whom we refer for a detailed discussion.

Remark

The pure *hyperelastic* localization tensor $\mathbf{q}_{el}$ becomes singular with eigendirection $\mathbf{n}$ as soon as

$$G[b_n^e+1]+L[1-\ln J_e]=0 \quad \rightsquigarrow \quad J_e=\exp\left(\frac{G}{L}[\min(b_n^e)+1]+1\right).$$

On the other hand, the *convex* domain of this widely used compressible extension is known to be limited by

$$U''=0 \rightsquigarrow J_e=\exp\left(\frac{G}{L}+1\right)\leqslant \exp\left(\frac{G}{L}[\min(b_n^e)+1]+1\right).$$

Thus, the first possibility for the *onset* of localization in the case of a mere *hyperelastic* response is encountered when the domain of applicability of this constitutive model is already exceeded. □

10. EXAMPLE: NEO-HOOKE/VON MISES MODEL

As an example we combine the *isotropic* compressible Neo–Hooke elastic model with the *isotropic* von Mises yield condition with *associated* flow rules

$$\Phi=|\operatorname{dev}\bar{\mathbf{T}}|-\sqrt{\tfrac{2}{3}}[Y_0+K]=\varphi-\sqrt{\tfrac{2}{3}}[Y_0+K]. \tag{86}$$

In accordance with the rules of tensor analysis a *covariant-contravariant* representation of the stress measure $\bar{\mathbf{T}}$ results in a *contravariant-covariant* representation of the flow direction $\bar{\mathbf{N}}$

$$\bar{\mathbf{N}}=\bar{\mathbf{M}}=\frac{\operatorname{dev}\bar{\mathbf{T}}^t}{\varphi} \quad \rightsquigarrow \quad \mathbf{v}=\boldsymbol{\mu}=\frac{\mathbf{s}}{\varphi} \quad \text{and} \quad \bar{N}=\bar{M}=\sqrt{\tfrac{2}{3}} \quad \text{with} \quad \mathbf{s}=\operatorname{dev}\boldsymbol{\tau}. \tag{87}$$

With $v_1=\mu_1=0$ the nondimensional function $Y(\mathbf{n})$, or rather $\tilde{Y}(\mathbf{n})$, for the determination of the critical hardening modulus H_{cr} is given by

$$\tilde{Y}(\mathbf{n})=\frac{\varphi^2}{2G^2}Y(\mathbf{n})=\frac{1}{b_n^e}[\mathbf{n}\cdot\mathbf{e}_e^2\cdot\mathbf{n}-\alpha(b_n^e)[\mathbf{n}\cdot\mathbf{e}_e\cdot\mathbf{n}]^2] \quad \text{with} \quad \mathbf{e}_e=\frac{1}{G}\mathbf{b}_e\cdot\mathbf{s}. \tag{88}$$

Parametrized in the principal axes $\bar{\mathbf{E}}_i$ of the elastic Finger tensor $\mathbf{b}_e$, the function $\tilde{Y}(\mathbf{n})$ is now expressed for a plane case in terms of the spatial angle θ

$$\tilde{Y}(\theta)=\frac{1}{b_n^e}[e_I^2n_I^2+e_{II}^2n_{II}^2-\alpha(b_n^e)[e_In_I^2+e_{II}n_{II}^2]^2] \quad \text{with} \quad \begin{bmatrix} n_I \\ n_{II} \end{bmatrix}=\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}. \tag{89}$$

The explicit solution for the extrema of $\tilde{Y}(\theta)$, as given by $\partial_\theta\tilde{Y}(\theta)=0$, i.e.,

$$\theta_{cr}=\arg\left(\max_\theta \tilde{Y}(\theta)\right) \tag{90}$$

is a fairly cumbersome expression. Although it can be readily obtained by a symbol manipulating software, it is not displayed here.

In the following examples we investigate the possibility for the *onset* of localization within a solid that is deforming homogeneously with *elasto-plastic* deformations according to the isotropic von Mises model in eqn 86. As noted earlier, the effect of the elastic part of the finite *elasto-plastic* straining, as contained in the elastic Finger tensor appearing implicitly in eqn 90, makes the explicit analysis of the critical band orientation and the corresponding critical hardening modulus cumbersome to carry out in practice. Hence, we display the results from expression 90 for some simple stress states. To this end, we consider

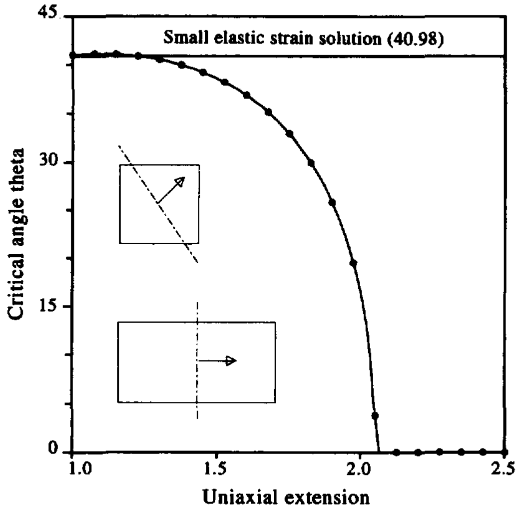

Fig. 2. Uniaxial extension: critical angle θ_{cr}.

the critical band orientation θ_{cr} defined with respect to the principal axes $\bar{\mathbf{E}}_i$ of $\mathbf{b}_e$ at the very *onset* of localization. In particular, we study how θ_{cr} depends on the actual stress state associated to the *elasto-plastic* deformation. Clearly, the actual stress state is determined by the *elastic* part $\mathbf{b}_e$ of the previous *elasto-plastic* straining. Therefore, it is not necessary to know the amount of *plastic* straining for our purpose since it leaves the determination of θ_{cr} and H_{cr} unaffected. The actual plastic deformation $\mathbf{F}_p$ is only needed to determine $\mathbf{F}_e$ from the given total deformation gradient $\mathbf{F}$ and to evaluate the actual hardening modulus $H(\kappa)$. Here we shall assume that we know the elastic part $\mathbf{b}_e$ of the *elasto-plastic* straining and thus the actual stress state beforehand. Thereby, as discussed before, the stress is connected by a one to one relation to $\mathbf{b}_e$ due to the *hyperelastic* part of the constitutive model. For the elastic constants contained in α as defined in eqn 78 we assume $G = 80.19$ GPa and $L = 110.75$ GPa. These values are typical for metals and have frequently been employed in the literature on computational plasticity, see, e.g., Simo (1992).

Uniaxial tension

For the case of uniaxial tension, we assume that the elastic deformation gradient $\mathbf{F}_e$ within an elasto-plastic deformation is given with respect to a fixed co-ordinate system $\mathbf{E}_i$ as

$$\mathbf{F}_e = \gamma_{ut}\mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3 \quad \text{with} \quad \gamma_{ut} \geqslant 1. \tag{91}$$

Clearly, the principal axes $\bar{\mathbf{E}}_i$ of $\mathbf{b}_e$ coincide with $\mathbf{E}_i$ in this case. The critical angle θ_{cr} at the possible *onset* of localization is plotted against the amount γ_{ut} in Fig. 2. It is noted that the inclination of a possible localization band starts from an orientation of 41°, as predicted by the small strain solution in Ottosen and Runesson (1991), and rotates with increasing deformation into an orientation that is perpendicular to the direction of extension, thus simulating a splitting tension failure type mode.

The critical hardening modulus H_{cr}, as computed from eqn 83, is displayed in Fig. 3. Remarkably, H_{cr} takes on a negative value for small elastic deformations. At finite elastic strains even a rather large positive $H \leqslant H_{cr}$ allows for the *onset* of localization, i.e., for the academic case that the yield limit allows for the development of very high stress levels or elastic strains, respectively, the *onset* of localization would be possible even if the material hardens extremely.

Pure shear

For the case of pure shear. we assume that the elastic deformation gradient $\mathbf{F}_e$ within an elasto-plastic deformation is given with respect to a fixed co-ordinate system $\mathbf{E}_i$ as

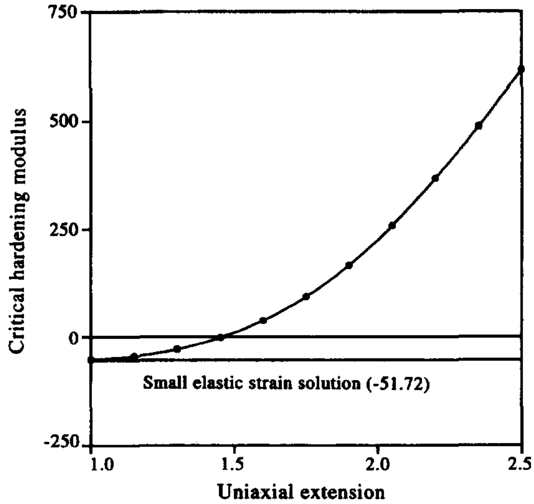

Fig. 3. Uniaxial extension: critical hardening modulus H_{cr}.

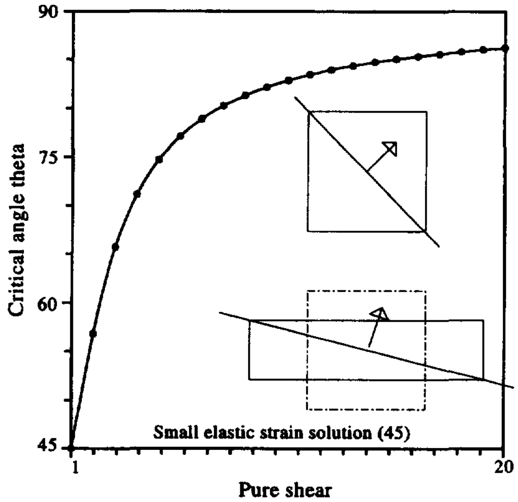

Fig. 4. Pure shear: critical angle θ_{cr}.

$$\mathbf{F}_e = \gamma_{ps}\mathbf{E}_1 \otimes \mathbf{E}_1 + \frac{1}{\gamma_{ps}}\mathbf{E}_2 \otimes \mathbf{E}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3 \quad \text{with} \quad \gamma_{ps} \geqslant 1. \tag{92}$$

Also, in this case, the principal axes $\bar{\mathbf{E}}_i$ of $\mathbf{b}_e$ coincide with $\mathbf{E}_i$. The critical angle θ_{cr} at the possible *onset* of localization is plotted against the amount γ_{ps} in Fig. 4. It is noted that the inclination of a possible localization band starts from an orientation of 0°, as predicted by the small strain solution in Ottosen and Runesson (1991), and rotates with increasing deformation into an orientation that is parallel to the direction of the major elastic principal strain. It appears that in this case, the orientation of the band at the possible *onset* of localization is convected with the deformation.

The critical hardening modulus H_{cr}, as computed from eqn 83, is displayed in Fig. 5. The small elastic strain solution in Ottosen and Runesson (1991) predicts $H_{cr} = 0$. At finite elastic strains even a tremendously large positive $H \leqslant H_{cr}$ allows for the *onset* of localization. Thus, as in the previous example, the sensitivity with respect to the *onset* of localization is increasing with elastic straining.

Simple shear

For the case of simple shear, we assume that the elastic deformation gradient $\mathbf{F}_e$ within an elasto-plastic deformation is given with respect to a fixed co-ordinate system $\mathbf{E}_i$ as

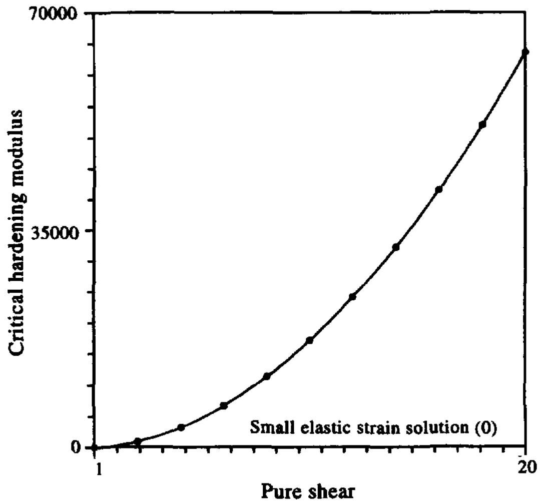

Fig. 5. Pure shear: critical hardening modulus H_{cr}.

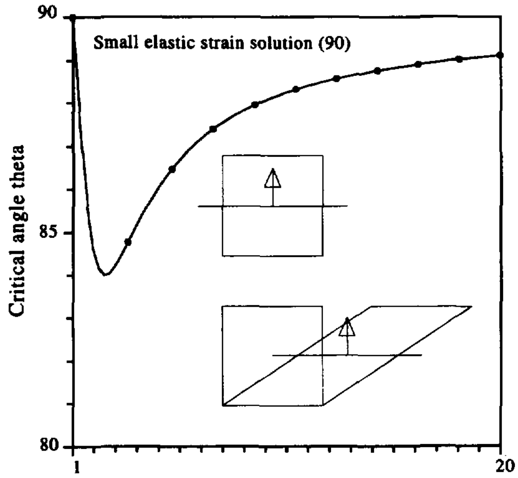

Fig. 6. Simple shear: critical angle θ_{cr}.

$$\mathbf{F}_e = \mathbf{I} + \gamma_{ss}\mathbf{E}_1 \otimes \mathbf{E}_2 \quad \leadsto \quad \mathbf{b}_e = \gamma_{ps}^2 \bar{\mathbf{E}}_1 \otimes \bar{\mathbf{E}}_1 + \frac{1}{\gamma_{ps}^2}\bar{\mathbf{E}}_2 \otimes \bar{\mathbf{E}}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3. \tag{93}$$

Expressed in Euler principal axes $\bar{\mathbf{E}}_i$, the associated elastic Finger tensor $\mathbf{b}_e$ has the same structure as for the case of pure shear in the previous example. The relation between γ_{ss} and γ_{ps} is given as $\gamma_{ss} = \gamma_{ps} - \gamma_{ps}^{-1} \geqslant 0$. Thus, for simple shear, the pure shear results in the previous example have to be augmented by the rotation of the principal axes of $\mathbf{b}_e$ to give a representation in the fixed co-ordinate axes $\mathbf{E}_i$. Expressed in γ_{ss}, this correction is given by $\tilde{\theta} = \arctan(2/\gamma_{ss})/2$, see, e.g., Ogden (1984). For the *onset* of localization the critical angle $\theta_{cr} + \tilde{\theta}$ with respect to $\mathbf{E}_1$ is plotted against the amount γ_{ss} in Fig. 6. The inclination of a possible localization band starts from an orientation of 90°, as predicted by the small strain solution in Ottosen and Runesson (1991), and rotates with increasing deformation into an orientation of $\approx$84°. Subsequently, the orientation of the band rotates back to very large amounts of shear into an inclination that simulates a *deck of cards* solution. Clearly, the critical hardening modulus H_{cr} develops in accordance with Fig. 5.

Remark

For small elastic strains, i.e., at the identity $\mathbf{F}_e = \mathbf{I} \leadsto \mathbf{b}_e = \mathbf{I}$ and $b_n^e = 1$ we retrieve the solution of the geometrically linear theory in Ottosen and Runesson (1991)

$$Y(\theta) = 2[v_I^2 n_I^2 + v_{II}^2 n_{II}^2 - \alpha[v_I n_I^2 + v_{II} n_{II}^2]^2] \quad \text{with} \quad \alpha = \frac{G+L}{2G+L}.$$

The critical angle θ_{cr} with respect to the principal axes of the Kirchhoff stress and the critical hardening modulus H_{cr} are then explicitly given by

$$\cos 2\theta_{cr} = [\alpha^{-1} - 1]\frac{v_I^2 - v_{II}^2}{[v_I - v_{II}]^2} \quad \text{and} \quad H_{cr} = \frac{3}{2}G\alpha^{-1}[v_I + v_{II}]^2 - 6Gv_I v_{II} - 3G.$$

In the case of uniaxial tension with $v_I \propto 2/3$, $v_{II} \propto -1/3$ we obtain the classical result

$$\cos 2\theta_{cr} = \frac{1}{3}\frac{G}{G+L} \rightsquigarrow \theta_{cr} \approx 41^\circ \quad \text{and} \quad H_{cr} = -\frac{G}{4}\frac{[2G+3L]}{[G+L]} = -51.72\,\text{GPa}.$$

Equivalently, the case of pure shear with $v_I \propto 1$, $v_{II} \propto -1$ renders $\theta_{cr} = 45^\circ$ and $H_{cr} = 0$. □

11. SUMMARY AND CONCLUSION

In this work we examined the large strain localization properties of *hyperelasto-plastic* materials which are based on the *multiplicative* decomposition of the deformation gradient for the case of *strong discontinuities.*

First, we derived an explicit expression for the spatial *hyperelasto-plastic* tangent operator for general *anisotropic* and *nonassociated* material behaviour. Thereby, the simple structure of the *hyperelasto-plastic* tangent operator, which is formally achieved by introducing *hyper-elastic* tangent relations for the rate of the Kirchhoff stress, resembles the structure of the geometrically linear theory.

Next, the *regularized discontinuous* spatial velocity gradient was derived and its interesting structure in terms of a rank one update was discussed in detail. With this prerequisite at hand, traction continuity in the spatial setting leads to the *localization* condition, or rather the *admissibility* condition for maintaining a regularized strong discontinuity. The exploitation of the particular structure of the tangent operator admits the localization condition in a format entirely similar to the geometrically linear analysis. In particular, special emphasis was placed on the loading conditions inside and outside an anticipated localization band. Thereby, the scenarios of elastic-elastic unloading and plastic–plastic loading render the classical localization condition in terms of a singular *hyperelastic* or *hyperelasto–plastic* localization tensor, respectively. The alternative assumption of plastic loading inside and elastic unloading outside the band requires a negative eigenvalue of the *hyperelasto-plastic* localization tensor.

Finally, based on a general representation for *isotropic* materials, examples were given for the case of *associated* von Mises flow rule. To this end, the critical localization direction and the critical hardening modulus at the *onset* of localization were investigated with respect to the amount of previous *elastic* straining or rather the actual stress state and compared to the analytical results of a small elastic strain solution. It is intriguing, that the determination of the critical localization direction and the critical hardening modulus is entirely unaffected by the amount of plastic straining. Moreover, the analysis of uniaxial extension, pure and simple shear revealed that, at finite elastic strains, even a rather large positive $H \leqslant H_{cr}$ allows for the *onset* of localization. Thereby, the corresponding spatial orientation may considerably depart from the small elastic strain solution.

In conclusion, it is believed that this work clarified issues pertaining to the formulation and analysis of *strong discontinuities* in the framework of *hyperelasto-plastic* materials at large strains and forms an indispensable prerequisite to future numerical developments.

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